GEOMETRIC THEOREMS AND PROBLEMS FOR HARMONIC MEASURE

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Abstract. We review some results and open problems for harmonic measure. Their common element is their simple geometric character. The classical such results are the projection estimates of Beurling, Nevanlinna, and Hall. Recent projection theorems are due to Marshall, Sundberg, Solynin and others. One of the methods of the proofs of these theorems is the method of extremal metrics and quadratic differentials. The classical symmetrization result for harmonic measure (originally proven by the star-function method of Baernstein) can now be proven by the polarization technique. There remain, however, several open problems. For most of these problems there are reasonable conjectures. The paper discusses the above results, problems, and methods. It contains several new open problems.

1. Introduction

Harmonic measure is one of the most important conformal invariants. It is a standard tool in complex analysis which provides a connection between potential theory and geometric function theory. The name harmonic measure was introduced by R. Nevanlinna in 1934 but methods related to harmonic measure had been used much earlier by Lindelf, Carleman, Ostrowski, F. and R. Nevanlinna.

Let $D$ be a domain in the complex plane $\mathbb{C}$ and let $E$ be a Borel set on the boundary $\partial D$ of $D$. The harmonic measure of $E$ at $z \in D$ relative to $D$ is the Perron solution $u(z)$ of the Dirichlet problem in $D$ with boundary values 1 on $E$ and 0 on $\partial D \setminus E$. More precisely, let $\chi_E = 1$ on $E$ and $\chi_E = 0$ on $\partial D \setminus E$. Then

$$u(z) = \sup \{ v(z) : v \text{ subharmonic in } D \text{ and } \limsup_{w \to \zeta} v(w) \leq \chi_E(\zeta) \text{ for } \zeta \in \partial D \}.$$ 

We will use the following notation for harmonic measure: Let $\Omega \subset \mathbb{C}$ be any open set and $K$ be a compact set in $\mathbb{C}$. $\omega(z, K, \Omega)$ will denote the harmonic measure at $z$ of the set $K \cap \text{clos} \Omega$ relative to the component of $\Omega \setminus K$ that contains $z$. We also set $\omega(z, K, \Omega) = 1$ for $z \in K \cap \text{clos} \Omega$ and $\omega(z, K, \Omega) = 0$ for those $z$ that lie in any component of $\Omega$ whose closure does not intersect $K$. The closures above are taken with respect to the topology of the extended complex plane. Some more notation: For $A \subset \mathbb{C}$, $-A = \{-z : z \in A\}$ and $\overline{A} = \{\bar{z} : z \in A\}$.

For fixed $z$ and $\Omega$, $\omega(z, K, \Omega)$, as a function of $K$, is a probability measure on $\text{clos} \Omega$. This probability measure has a beautiful and important probabilistic interpretation: Let $B_t$, $t > 0$ be Brownian motion on the plane starting from $z$. Let $\tau = \inf \{ t > 0 : B_t \notin \Omega \setminus K \}$ be the first exit time of $B_t$ from $\Omega \setminus K$. Then $\omega(z, K, \Omega)$ is the probability that $B_\tau \in K$. Powerful probabilistic techniques in function theory are based on this probabilistic interpretation.

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For the main properties of harmonic measure and some of its applications we refer to [57], [42], [71], [34], [33]. For more details on its probabilistic interpretation see [61]. For questions related to the definition of harmonic measure and its conformal invariance see [1].

In this paper we survey results on harmonic measure that have a simple geometric character. We study the behaviour of harmonic measure under geometric transformations such as projection, symmetrization, and polarization, and we review some extremal problems. We also present several open problems. It is not our aim to survey all results about harmonic measure. Several important topics are omitted. In the last section we comment on our main omissions and offer references.

2. Projection theorems and problems

The harmonic measure can be explicitly computed only for a few pairs of $K$ and $D$ with the use of conformal maps. Therefore it is of importance to find estimates for it in terms of majorization principles or monotonicity results.

**Theorem 2.1 (The Beurling-Nevanlinna projection theorem).** Let $K$ be a closed set in the unit disk $D$. Let $K_1 = \{-|z| : z \in K\}$. Then

\[ \omega(z, K, D) \geq \omega(|z|, K_1, D), \forall z \in D. \]

$K_1$ is the circular projection of $K$ on the negative radius $[-1, 0]$ of the unit disk. Theorem 2.1 was proven independently by A.Beurling[19] and R.Nevanlinna[55] in 1933. It can be generalized in various ways (see [2], [58], [16]). Here is one of its extensions:

**Theorem 2.2 ([16]).** Let $y \in (0, 1)$ and let $K, K_1$ be as in Theorem 2.1. Then

\[ \omega(iy, \partial D \setminus K) + \omega(-iy, \partial D \setminus K) \leq \omega(iy, \partial D \setminus K_1) + \omega(-iy, \partial D \setminus K_1). \]

Beurling[19] proved also a related result which implies the following:

**Theorem 2.3 (Beurling’s shove theorem).** Let $K_1$ be the union of a finite number of closed intervals on the radius $[-1, 0]$ of the unit disk. Let $l$ be the total length of $K_1$. Then

\[ \omega(0, K_1, D) \geq \omega(0, [-1, -1+l], D) \]

The basic tool in the proof of Theorems 2.1 and 2.3 is F.Riesz’s fundamental representation theorem for superharmonic functions. $\omega(\cdot, K, \mathbb{D})$ (or $\omega(\cdot, K_1, \mathbb{D})$) is a superharmonic function on $\mathbb{D}$. So, by Riesz’s theorem, it is the potential of a measure (Riesz mass) on $K$ (or on $K_1$). We construct a new potential by sweeping suitably the Riesz mass on $K_1$ (or on $[-1, -1+l]$). Then (2.1) (or (2.3)) follows from the comparison of the two potentials with the help of the maximum principle.

A.Yu.Solynin[68] proved the following generalization of Beurling’s shove theorem.

**Theorem 2.4 (Solynin).** Let $K \subset [-1, 1]$ be the union of a finite number of closed intervals of total length $2l$. Let $K' = [-1, -1+l] \cup [1-l, 1]$. Then

\[ \omega(0, K, \mathbb{D}) \geq \omega(0, K', \mathbb{D}). \]

This theorem had been conjectured by S.Segawa[64]. By using the method of sweeping the Riesz mass, M.Essén and K.Haliste[27] had proved Segawa’s conjecture with the additional assumption that $-K = K$, i.e. when $K$ is symmetric with respect to the imaginary axis. Solynin’s proof of Theorem 2.4 uses that result of Essén and Haliste. Some other extensions of Beurling’s shove theorem have been proved in [62] and [18].
We present now two conjectures related to Beurling’s shove theorem. The first would be a nice counterpart for Theorem 2.2. The second implies Theorem 2.4.

**Conjecture 1.** Let \( K \subset (0, 1) \) be the union of a finite number of intervals of total length \( l \). Let \( K^* = [1 - l, 1] \). Then for \( y \in (-1, 1) \) we have

\[
\omega(iy, K, D) \geq \omega(iy, K^*, D).
\]

**Conjecture 2** ([14]). Let \( K_1 \subset [-1, 0) \) be a finite union of closed intervals of total length \( m_1 \). Let \( K_2 \subset (0, 1] \) be a finite union of closed intervals of total length \( m_2 \). Then

\[
\omega(0, K_1 \cup K_2, D) \geq \omega(0, K_1' \cup K_2', D),
\]

where \( K_1' = [-1, -1 + m_1] \) and \( K_2' = [1 - m_2, 1] \).

In the upper-half-plane \( \mathbb{H} \) the classical projection theorem is due to T.Hall[38]:

**Theorem 2.5 (Hall’s lemma).** Let \( E \subset \mathbb{H} \) be a closed set and \( E_1 = \{-|z| : z \in E\} \) be its circular projection on the negative real axis. Then there exists an absolute constant \( k, \frac{2}{3} < k \leq 1 \), such that

\[
\omega(x + iy, E, \mathbb{H}) \geq k\omega(-|x| + iy, E_1, \mathbb{H}).
\]

The proof of Hall’s lemma uses an elaboration of the sweeping of the Riesz mass. W.K. Hayman constructed an example which shows that \( k < 1 \) (see [40], [53]). On the other hand D.Gaier[32] proved that if \( E \) is a single arc with one end-point at the origin then (2.7) is valid with \( k = 1 \). J.A.Jenkins[47] proved a generalization of Gaier’s result which takes into account the location of the other end point of the arc. Finally, Solynin[69] proved that (2.7) holds with \( k = 1 \) for every continuum \( E \subset \mathbb{H} \). The problem of determining the exact value of the best constant in Hall’s lemma is open.

We review now some recent projection theorems.

**Theorem 2.6 (FitzGerald-Rodin-Warschawski[30], Solynin[65]).** Let \( E \) be a continuum in \( \text{clos} \mathbb{D} \) and let \( d \) be the diameter of \( E \). Let \( E_d \) be an arc on \( \partial \mathbb{D} \) of diameter \( d \). Then

\[
\omega(0, E, \mathbb{D}) \geq \omega(0, E_d, \mathbb{D}).
\]

This theorem was also proved by Jenkins[48].

For the radial projection of curves in \( \mathbb{D} \) the definitive result is the following:

**Theorem 2.7 (Marshall-Sundberg[54]).** Let \( \mathcal{E} \) be the family of all continua \( E \in \text{clos} \mathbb{D} \). For \( E \in \mathcal{E} \), let \( \hat{E} = \{z/|z| : z \in E\} \) be the radial projection on \( \partial \mathbb{D} \) and \( l_E \) be the length of \( \hat{E} \).

(a) Let \( C \approx 0.997 \) be the harmonic measure of the two long sides of a 3 : 1 rectangle evaluated at its center. Then for all \( E \in \mathcal{E} \),

\[
\omega(0, E, \mathbb{D}) \geq C\omega(0, \hat{E}, \mathbb{D})
\]
and \( C \) is the largest constant for which (2.9) holds for all \( E \in \mathcal{E} \).

(b) There is an absolute constant \( K \approx 2\pi \frac{350}{360} \) such that if \( E \in \mathcal{E} \) with \( l_E \leq K \) then

\[
\omega(0, E, \mathbb{D}) \geq \omega(0, \tilde{E}, \mathbb{D})
\]

and for every \( l \) with \( K < l \leq 2\pi \), there exists \( E \in \mathcal{E} \) with \( l_E = l \) for which \( \omega(0, E, \mathbb{D}) < \omega(0, \tilde{E}, \mathbb{D}) \).

Finally a result which involves orthogonal projection:

**Theorem 2.8 (Solynin[69]).** Let \( E \) be a continuum in \( \text{clos} \mathbb{H} \) and let \( E^* = \{ \Re z : z \in E \} \). Assume that every curve \( \alpha \subset \{ z \in \mathbb{H} : \Re z \in E^* \} \) joining \( z_o \in \mathbb{H} \) and \( E^* \) intersects \( E \). Then

\[
\omega(z_o, E, \mathbb{H}) \geq \omega(z_o, E^*, \mathbb{H}).
\]

For the proof of the above results Jenkins, Marshall, Sundberg and Solynin used the method of extremal length. Relations between extremal length and harmonic measure have been studied by A.Beurling[19], and J.Hersch[44]. It was Jenkins who systematically used extremal length for the solution of numerous problems for harmonic measure and other conformal invariants (see the references in [69]). Here is a brief description of Jenkins’s method: The relation between extremal length and the harmonic measure of a continuum (the so-called *Lemma on the module of a triad* [46]) reduces extremal problems for harmonic measure to extremal problems for moduli of doubly-connected domains. One can then attack the latter problems using Jenkins’s theory for extremal moduli which describes the extremal domains with the help of quadratic differentials, see [45], [49], [50], [54], [69]. This method has been proved to be very powerful and successful. It has, however, an important limitation in its application to problems for harmonic measure: It can be applied only to problems involving harmonic measure of a *continuum*. The following problem is a special case of a problem of W.H.J.Fuchs ([12], problem 3.22).

**Problem 1.** Find \( \inf \{ \omega(0, \gamma_1 \cup \gamma_2, \mathbb{D}) : \gamma_1, \gamma_2 \text{ are curves in } \mathbb{D} \text{ such that } \gamma_1 \cup \gamma_2 \text{ meets every radius of } \mathbb{D} \} \).

3. Symmetrization

There exist many types of symmetrization. Here we deal only with Steiner and circular symmetrization. Let \( O \) be an open set in the plane. The Steiner symmetrization of \( O \) with respect to the real axis is the transformation of \( O \) into a symmetric set \( SO \) defined as follows:

\[
SO = \{ z = x + iy : x \in \mathbb{R}, 2|y| < l(x, O) \},
\]

where \( l(x, O) \) is the linear Lebesgue measure of \( O \cap \{ z : \Re z = x \} \).

For a compact set \( E \subset \mathbb{C} \) the Steiner symmetrization is

\[
SE = \{ z = x + iy : E \cap \{ \Re z = x \} \neq \emptyset, 2|y| \leq l(x, E) \}.
\]

The circular symmetrizations \( SO \) and \( SE \) with respect to the positive semi-axis \( \mathbb{R}^+ \) are

\[
SO = \{ z = re^{i\theta} : O \cap T_r \neq \emptyset, 2|\theta| < s(r, O) \} \cup \{ -r : O \cap T_r = T_r \}
\]

\[
SE = \{ z = re^{i\theta} : E \cap T_r \neq \emptyset, 2|\theta| \leq s(r, E) \},
\]

where \( T_r = \{ z : |z| = r \} \) and \( s(r, O) \) (resp. \( s(r, E) \)) is the angular Lebesgue measure of \( O \cap T_r \) (resp. of \( E \cap T_r \)), \( r \geq 0 \).
The classical symmetrization result for harmonic measure is due to A. Baernstein [2] who improved an earlier result of K. Haliste [36].

**Theorem 3.1 (Baernstein).** Let $D$ be a domain with $D \subset \mathbb{D}$ and let $\alpha = \partial D \cap \partial \mathbb{D}$. Then for all $r \in (0,1)$ and all increasing convex functions $\Phi : \mathbb{R} \to \mathbb{R}$

$$
\int_0^{2\pi} \Phi(\omega(re^{i\theta}, \alpha, D)) \, d\theta \leq \int_0^{2\pi} \Phi(\omega(re^{i\theta}, S\alpha, SD)) \, d\theta.
$$

Baernstein proved this theorem using the star-function method which can also give the analogous result for Steiner symmetrization. M. Essén and D. Shea [28] proved an equality statement for Theorem 3.1.

![Figure 3: D, E and their Steiner symmetrizations SD and SE.](image)

The star-function method is used for the solution of certain extremal problems for which the competing functions $v$ are subharmonic and the expected extremal function $u$ is (essentially unique and) harmonic in a symmetric region. For each $v$, Baernstein defined a certain maximal function $v^*$ (the star-function of $v$) and showed that the solution of the extremal problem is reduced to the inequality $v^* \leq u^*$. The heart of the method is the fact that $v^*$ remains subharmonic, while the symmetry of the (expected) extremal domain implies that $u^*$ is harmonic. It follows that the function $v^* - u^*$ is subharmonic and therefore, in order to prove the desired inequality $v^* - u^* \leq 0$, one can use the maximum principle. A detailed account is in [3], [9], or [10].

Baernstein [8] and Fryntov [31] developed a variant of the star-function method which led Solynin [66] to prove the following theorem, conjectured by V. N. Dubinin [25].

**Theorem 3.2 (Solynin).** Let $D$ be a domain in $\{z : \Re z < 0\}$ and let $E \subset \partial D$ be a segment on the imaginary axis. Assume that the length of any segment lying in $D$ parallel to the imaginary axis does not exceed $s > 0$. Then for any $z \in D$

$$
\omega(z, E, D) \leq \omega(\Re z, SE, \{z : \Re z < 0, |\Im z| < \frac{s}{2}\}).
$$

Solynin proved also an integral inequality like (3.4) with an equality statement.

For more information about symmetrization we refer to [2], [3], [9], [10], [25], [39], [41]. The paper [63] of A. R. Pruss contains some interesting results and open problems related to harmonic measure and symmetrization.
4. Polarization

Polarization with respect to the real axis is a geometric transformation that preserves the symmetric part of a set and moves the non-symmetric part to the upper half plane. To give the precise definition we need the following notation: $\mathbb{C}_+$ is the upper half plane and $\mathbb{C}_-$ is the lower half plane. If $F$ is a closed set in $\mathbb{C}$, then $F_+ = \text{clos } \mathbb{C}_+ \cap F$ and $F_- = \text{clos } \mathbb{C}_- \cap F$. If $O$ is an open set in $\mathbb{C}$, then $O_+ = \mathbb{C}_+ \cap O$ and $O_- = \mathbb{C}_- \cap O$.

The polarization $P_A$ of a closed or open set $A \subset \mathbb{C}$ is defined as follows: If $z, \bar{z} \in A$ then $z, \bar{z} \in P_A$. If neither $z, \bar{z} \in A$ then neither $z, \bar{z} \in P_A$. If exactly one of $z = x + iy$, $\bar{z}$ belongs to $A$ then $x + i|y| \in P_A$ and $x - i|y| \notin P_A(A)$. Below there is an equivalent definition.

**Definition 4.1.** Let $A$ be a closed or open set. The polarization $P_A$ of $A$ (with respect to $\mathbb{R}$) is $P_A = (A \cup \overline{A})_+ \cup (A \cap \overline{A})_-$. We define also the polarization $P_l A$ of $A$ with respect to any oriented line $l$.

**Definition 4.2.** Let $l$ be an oriented line and let $T_l : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the Möbius transformation that maps $\infty$ to $\infty$ and $l$ to $\mathbb{R}$ preserving their orientation. The polarization $P_l A$ of $A$ with respect to $l$ is $P_A = T_l^{-1} P T_l A$.

![Figure 4: A domain $D$ and its polarization $P_D$.](image)

Polarization was discovered by V.Wolontis[72] who studied the behaviour of certain extremal lengths under polarization. However, the idea of polarization can be traced in the proof of Theorem 2.3 in Beurling’s dissertation (see [19]). The main polarization result for harmonic measure is due to Solynin[67] and Betsakos[13],[15] who improved an earlier result of Øksendal[58] and Baernstein[29].

Let $D$ be a domain in $\mathbb{C}$, regular for the Dirichlet problem. Let $E \subset \partial D$ be a closed set and assume that $E$ satisfies the condition:

\[(4.1) \quad \overline{E} \cap D = \emptyset.\]

For the equality statements below it is assumed, in addition, that $D$ is bounded by a finite number of curves or arcs and $E$ consists of a finite number of curves or arcs.

We give the names (a), (b), (c) to the following conditions: (a) $E = P E$, $D = P D$. (b) $E = P \overline{E}$, $D = P \overline{D}$. (c) $D = \overline{D}$.

**Theorem 4.3 ([67], [15]).** With the above notation we have

\[(4.2) \quad \omega(x, E, D) \leq \omega(x, P E, P D), \quad x \in \mathbb{R} \cap D.\]

Equality holds in (4.2) for some $x \in \mathbb{R} \cap D$ if and only if (a) or (b) or (c) holds.

\[(4.3) \quad \omega(\bar{z}, E, D) \leq \omega(z, P E, P D), \quad z \in \overline{D}.\]
Equality holds in (4.3) for some \( z \in \overline{D} \) if and only if (b) holds.

(4.4) \( \omega(z, E, D) \leq \omega(z, \overline{P}E, \overline{P}D), \quad z \in D_+ \).

Equality holds in (4.4) for some \( z \in D_+ \) if and only if (a) holds.

(4.5) \( \omega(z, E, D) + \omega(\overline{z}, E, D) \leq \omega(z, \overline{P}E, \overline{P}D) + \omega(\overline{z}, \overline{P}E, \overline{P}D), \quad z \in D \).

Equality holds in (4.5) for some \( z \in D \) if and only if (a) or (b) or (c) holds.

(4.6) \( |\omega(z, E, D) - \omega(\overline{z}, E, D)| \leq \omega(z, \overline{P}E, \overline{P}D) - \omega(\overline{z}, \overline{P}E, \overline{P}D), \quad z \in D_+ \).

Equality holds in (4.6) for some \( z \in D_+ \) if and only if (a) or (b) or (c) holds.

For all \( x \in \mathbb{R} \) and all increasing convex functions \( \Phi : \mathbb{R} \to \mathbb{R} \)

(4.7) \[
\int_{D_x} \Phi(x + iy, E, D) \, dy \leq \int_{\overline{P}D_x} \Phi(x + iy, \overline{P}E, \overline{P}D) \, dy,
\]
where \( D_x = \{ y : x + iy \in D \} \) and \( \overline{P}D_x = \{ y : x + iy \in \overline{P}D \} \).

Equality holds in (4.7) for some \( x \in \mathbb{R} \cap D \) and some nonconstant, increasing, convex function \( \Phi \) if and only if (a) or (b) or (c) holds.

Figure 5: An illustration for Theorem 4.3.

Oksendal [58] (see also Baernstein’s lemma in [29]) proved (4.2) with the additional assumptions that \( D = \overline{D} \) and \( E \cap \overline{E} = \emptyset \). The basic inequality in Theorem 4.3 is (4.2). The other inequalities follow easily from (4.2) by the maximum principle. (4.7) follows from (4.5) and a convexity argument. The proof of (4.2) in [67] is based only on the maximum principle applied to suitably chosen functions and domains. The proof of (4.2) in [13] uses a technique of Oksendal [58] which is based on Brownian motion and gives also some results on Brownian motion. We present one of them:

Let \( D \) and \( E \) be as in theorem 4.3. Let \( \tau_D = \inf \{ t > 0 : B_t \notin D \} \) be the first exit time from \( D \). For \( z \in D \) and \( 0 \leq \tau_1 < \tau_2 \leq +\infty \), let

(4.8) \[
\omega_{\tau_2}^{\tau_1}(z, E, D) = P^z(B_{\tau_1} \in E, \, \tau_1 < \tau_D < \tau_2).
\]

Thus \( \omega_{\tau_1}^{\tau_2}(z, E, D) \) is the probability that a Brownian motion starting at \( z \) exits \( D \) through \( E \), in the time interval \((\tau_1, \tau_2)\). For \( \tau_1 = 0 \) and \( \tau_2 = +\infty \), \( \omega_{\tau_1}^{\tau_2}(z, E, D) \) is the usual harmonic measure \( \omega(z, E, D) \).

**Theorem 4.4** ([15]). With the above notation we have

(4.9) \[
\omega_{\tau_1}^{\tau_2}(x, E, D) \leq \omega_{\tau_1}^{\tau_2}(x, \overline{P}E, \overline{P}D), \quad x \in \mathbb{R} \cap D.
\]

Polarization is a transformation simpler than symmetrization in the sense that it does not change the shape of a set as drastically as symmetrization does. For an illustration see the figure below.
Figure 6: $D$ is a domain with a slit $K$ and $E$ is a symmetric closed set on $\partial D$. $SD = D \cup K$ (the slit disappears). $P_1D$ has a slit $K'$ which is further from $x \in \mathbb{R}$ than $K$ is. Symmetrization yields a trivial inequality while polarization gives the nontrivial result $\omega(x,E,D) \leq \omega(x,E,P_1D)$.

Wolontis[72] made the important observation that circular symmetrization results can be proved by the successive application of polarizations with respect to suitable axes. This idea has been used in various ways for the proof of rearrangement inequalities, symmetrization results for the capacity of condensers, and projection estimates for harmonic measure, see [11], [25], [58] respectively. Dubinin[25] used this idea in the following way: Let $D$ be a domain in the plane. Assume first that $\partial D$ is the union of a finite number of Jordan curves each consisting of a finite number of horizontal and vertical segments. By drawing a finite number of vertical lines we divide $D$ into a finite number of rectangles. Applying then successively a finite number of polarizations with respect to suitable lines we symmetrize one rectangle after the other. At the end we obtain the Steiner symmetrization of $D$. For general $D$, we use an approximation argument. The figure below shows a simple application of this technique.

Figure 7: The Steiner symmetrization $SD$ of the domain $D$ is obtained by the successive application of two polarizations, first with respect to the oriented line $l$ and then with respect to the oriented line $L$: $SD = P_lP_1D$.

Øksendal[58] used a similar technique to approximate circular symmetrization. Let $K \subset \mathbb{D}$. By applying successive polarizations first with respect to the line $\{re^{i\theta} : r \in \mathbb{R}, \theta = \pi/2^n\}$ and then with respect to the line $\{re^{i\theta} : r \in \mathbb{R}, \theta = -\pi/2^n\}$ (at the $n$'th step) we obtain (in the limit) the circular symmetrization of $K$. Øksendal used, however, his reflection lemma and obtained only the Beurling-Nevanlinna projection theorem (Theorem 2.1). If we use the stronger polarization result (Theorem 4.3), we obtain Theorem 3.1.

The fact that Steiner and circular symmetrizations can be approximated by a sequence of polarizations leads to simple proofs of symmetrization results originally proven by Baernstein’s star-function method. In particular, equality statements for symmetrization inequalities are proven easily via polarization. On the other hand the star-function method not only proves symmetrization results but also, more generally, solves extremal problems for which the extremal domains are symmetric, see [2], [3], [52].

The following conjecture compares the effect of polarization (or symmetrization) on harmonic measure with that of domain expansion.
Let $K \subset \mathbb{D}$ be a compact set and let $\mathbb{D}_r = \{ z : |z| < r \}$, $r > 1$. Then
\begin{align}
\omega(0, K, \mathbb{D}) - \omega(0, PK, \mathbb{D}) \geq \omega(0, K, \mathbb{D}_r) - \omega(0, PK, \mathbb{D}_r),
\end{align}
\begin{align}
\omega(0, K, \mathbb{D}) - \omega(0, SK, \mathbb{D}) \geq \omega(0, K, \mathbb{D}_r) - \omega(0, SK, \mathbb{D}_r).
\end{align}

5. Harmonic measure of radial slits

Suppose that $K$ is a compact set in $(0, 1]$ and let $0 \leq \theta_1 < \theta_2 < \ldots < \theta_n < 2\pi$. Consider the unions of radial slits $E, E^*$ on the unit disk defined by
\begin{align*}
E_K &= \bigcup_{k=1}^n \exp(i\theta_k)K, \quad E^*_K = \bigcup_{k=1}^n \exp(2\pi i(k - 1)/n)K.
\end{align*}

Dubinin[24] invented the method of dissymmetrization for the proof of the following theorem which had been conjectured by A.A.Gonchar (see [23] problem 7.45).

**Theorem 5.1 (Dubinin).** Assume that $K = [\rho, 1]$, $\rho > 0$. Then
\begin{align}
\omega(0, E_K, \mathbb{D}) \leq \omega(0, E^*_K, \mathbb{D}).
\end{align}
with equality only if $E_K = e^{i\phi}E^*_K$, for some $\phi \in \mathbb{R}$.

Baernstein[5] generalized Gonchar’s problem as follows:

**Conjecture 4 (Baernstein).** For all compact sets $K \subset (0, 1]$, 
\begin{align}
\omega(0, E_K, \mathbb{D}) \leq \omega(0, E^*_K, \mathbb{D}),
\end{align}
and more generally
\begin{align}
\int_0^{2\pi} \Phi(\omega(te^{i\psi}, E_K, \mathbb{D}))d\psi \leq \int_0^{2\pi} \Phi(\omega(te^{i\psi}, E^*_K, \mathbb{D}))d\psi,
\end{align}
for all $t \in (0, 1)$ and all increasing convex functions $\Phi : \mathbb{R} \to \mathbb{R}$.

A similar conjecture was also published by Haliste[37]. Baernstein[6] proved (5.3) for $n = 2$ and $n = 3$. He used dissymmetrization and the star-function method. Recently Solynin[70] proved (5.2) under the assumption that $K$ is a closed interval. His proof combines the method of extremal metric with a new version of dissymmetrization on Riemann surfaces, and involves computations with elliptic theta functions. Haliste[37] also used dissymmetrization to prove a theorem for the harmonic measure of circular slits.

Dissymmetrization has been particularly successful for problems where the extremal domains have $n$-fold symmetry. It has been applied to extremal problems for various conformal invariants. The main idea is the following: Suppose that $f : \partial \mathbb{D} \to \mathbb{R}$ is a Lipschitz function with $n$-fold symmetry with respect to the points $\exp(i\theta^*) = \exp(2\pi i(k - 1)/n)$, $k = 2, 3, \ldots, n$. The dissymmetrization of $f$ is a certain rearrangement of $f$ with the properties: $g$ is Lipschitz and $g(\exp(i\theta_k)) = f(\exp(i\theta^*_k))$. Dubinin’s main achievement was the description of a specific procedure for the construction of $g$ and the proof that dissymmetrization preserves the Dirichlet integral. The papers [24], [25], [5], [37] contain accounts of dissymmetrization and additional references.

6. More research problems and conjectures

1. 

**Problem 2 (Solynin[69]).** Let $a \in (0, 1)$ and let $\mathcal{F}$ be the family of all simply connected domains $D \subset \mathbb{D}$ with $-a \notin D$, $a \notin D$. Find
\begin{align*}
\sup \{ \omega(0, \partial D, D) : D \in \mathcal{F} \}.
\end{align*}
2. Let \( D \subset \mathbb{C} \) be a domain with \( 0 \in D \). For \( R > 0 \), we set
\[
\omega_D(R) = \omega(0, \partial D \cap \{z : |z| \geq R\}, D) \quad \text{and} \quad \hat{\omega}_D(R) = \omega(0, \{z : |z| = R\}, D).
\]
The behaviour of the functions \( \omega_D \) and \( \hat{\omega}_D \) near \( \infty \) determines (in some sense) how large \( D \) is (see e.g. [71] p.112, [43], [26], [17]). For example, M. Essén [26] proved that: Every analytic function \( f : D \to D \) belongs to the Hardy space \( H^p \) (for some \( p \) that depends on \( f \)) if and only if \( \hat{\omega}_D(R) \leq CR^{-q} \), for all \( R \geq 1 \), where \( C, q \) are positive constants that depend only on \( f \). A domain \( D \) that has the property described in Essén’s result is called a Hardy domain.

**Problem 3.** Find a (euclidean) geometric condition that characterizes Hardy domains.

Note that for Bloch domains, \( BMO \) domains, and domains associated to the Nevanlinna class such geometric characterizations have been found (see [4], [57]).

It is not known whether the functions \( \omega_D \) and \( \hat{\omega}_D \) have always the same behaviour when \( R \) is near \( \infty \).

**Conjecture 5.** There exists a positive constant \( C > 0 \) such that for all domains \( D \) with \( 0 \in D \) and all \( R > 0 \), we have
\[
\omega_D(R) \geq C\hat{\omega}_D(R). \tag{6.3}
\]

Perhaps this problem becomes easier if we pose some additional hypothesis on \( D \) (starlike, simply connected, \( BMO \) domain, etc).

Next we present another problem for \( \omega_D(R) \). Let \( B \) be the family of all simply connected domains \( D \subset \mathbb{C} \) that have the properties: (a) there is no disk of radius larger than 1 contained in \( D \) and (b) \( 0 \in D \). It is obvious that if \( D \in B \) then \( \omega_D \) is a decreasing function of \( R \). In fact, one can prove that \( \omega_D \) decreases exponentially. This follows, at least intuitively, from the probabilistic interpretation of harmonic measure as hitting probability of Brownian motion in \( D \): A Brownian particle starting from the circle \( |z| = r \) and stopping when it hits the boundary of \( D \) has small probability to reach the circle \( |z| = r + 2 \), because \( D \in B \). Now repeated applications of the Markov property show that \( \omega_D \) decays exponentially: \( \omega_D(R) \leq Ce^{-\beta R} \), \( D \in B, R > 0 \) with absolute constants \( \beta, C \), see [17]. Define
\[
\beta(D) = \sup\{\beta > 0 : \exists C > 0 \text{ such that } \forall R > 0, \omega_D(R) \leq Ce^{-\beta R}\}.
\]

**Problem 4.** Find the exact value of the number
\[
\beta_0 = \inf_{D \in B} \beta(D). \tag{6.5}
\]

This problem, posed by C.Bishop [21], is the analog for harmonic measure of the Bloch constant problem. Bishop conjectured that \( \beta_0 = \beta(S) = \pi/2 \), where \( S \) is a strip of width 2. Bishop’s conjecture is false. In fact it is proved in [17] that
\[
1.1417 \leq \beta_0 \leq 0.4286\pi \approx 1.3458 \tag{6.6}
\]

The upper bound comes from the study of the **comb-domain**
\[
D_o = \mathbb{C} \setminus \left( \bigcup_{k \in \mathbb{Z}} \{2kx_1 + iy : y \geq y_1\} \cup \bigcup_{k \in \mathbb{Z}} \{(2k+1)x_1 + iy : y \leq -y_1\} \right),
\]
where \( x_1 \approx 0.66 \) and \( y_1 = \frac{1}{2}(1 + \sqrt{1 - x_1^2}) \) (\( y_1 \) is chosen so that \( D_o \in B \)). Perhaps \( D_o \) is an extremal domain for Bishop’s problem. We note, however, that it is not known even whether an extremal domain exists.
3. The next problem grew out of discussions with R.Bañuelos and A.Baernstein.

**Problem 5.** Let $B$ be as in Problem 4 and let $B_2 = \{ D \in B : D \subset \{ z : |z| < 2 \} \}$. Find
\[
\sup \{ \omega(0, \partial D \cap \{ z : |z| = 2 \}, D) : D \in B_2 \}.
\]

4. Most of the problems we presented so far can be formulated as extremal problems for $\omega(z, E, D)$, where $E, D$ satisfy specific euclidean geometric conditions. Another class of problems is created when we pose “conformal conditions” on $E$ or $D$. As an example of such problem we present a theorem which follows from the work of H.Grötzsch[35].

**Theorem 6.1.** Let $K$ be a continuum in $\mathbb{D}$ with $0 \in K$. Suppose that
\[
caph(K) = caph([0, r_o]),
\]
where $r_o \in (0, 1)$. Then
\[
\omega(re^{i\theta}, K, \mathbb{D}) \geq \omega(-r, [0, r_o], \mathbb{D}),
\]
for all $r \in (0, 1)$ and all $\theta \in [0, 2\pi)$.

Here $caph$ denotes hyperbolic capacity (see [71]). This theorem has been strengthened in [2].

Another problem with conformal conditions is the following: Let $K = \{ K \subset \mathbb{D} : K$ continuum, $\frac{1}{2} \in K, \omega(0, K, \mathbb{D}) = 0.1 \}$. Let $\mathbb{D}_2 = \{ z : |z| < 2 \}$. Clearly there exist unique numbers $m, M$ such that $[m, \frac{1}{2}] \in K$ and $[\frac{1}{2}, M] \in K$.

**Conjecture 6.**
\[
\min \{ \omega(0, K, \mathbb{D}_2) : K \in K \} = \omega(0, [m, \frac{1}{2}], \mathbb{D}_2),
\]
\[
\max \{ \omega(0, K, \mathbb{D}_2) : K \in K \} = \omega(0, [\frac{1}{2}, M], \mathbb{D}_2).
\]

7. **Concluding remarks**

We did not mention anything about higher dimensional analogs of the above theorems and problems. We refer to [27], [58], [73] for projection problems and to [11], [9] for symmetrization. The polarization results hold in all dimensions with essentially the same proofs.

The methods that we briefly discussed in this paper (extremal length-quadratic differentials, transport of the Riesz mass, polarization, symmetrization, star-function) can be applied to many kinds of problems in function theory, potential theory and partial differential equations (at least). It is impossible to give detailed references here. The interested reader can look at the surveys of Baernstein[3],[7],[9],[10], Brock and Solynin[22], Dubinin[25], and Kuz’mina[51].

Another important subject that we did not mention is that of the harmonic measure on domains with complicated boundary. There exist deep results of C.Bishop, L.Carleson, P.Jones, N.Makarov, T.Wolff and others on the support of the harmonic measure, and the distortion and boundary behaviour of conformal maps. Several references and open problems appear in C.Pommerenke’s book [60] and in Bishop’s collection of problems [20].

For the role played by harmonic measure in results related to the Hayman-Wu theorem we refer to [29], [33], [59] and the references therein.
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References

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